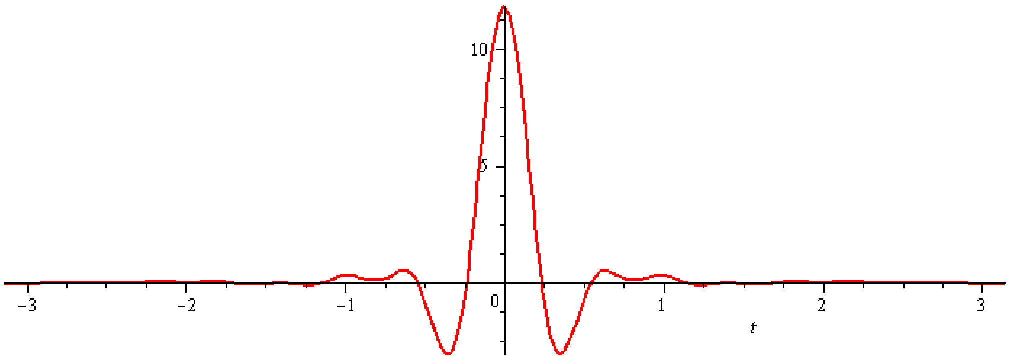
Frames, Quadratures and Global Illumination: New Math for Games



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WARNING

- This talk is **MATH HEAVY**
- We assume you understand the basics of:
  - Linear Algebra, Calculus, 3D Mathematics
  - Spherical Harmonic Lighting, Visibility, BRDF, Cosine Term
  - Monte Carlo Integration, Unbiased Spherical Sampling
  - Precomputed Radiance Transfer, Rendering Equation

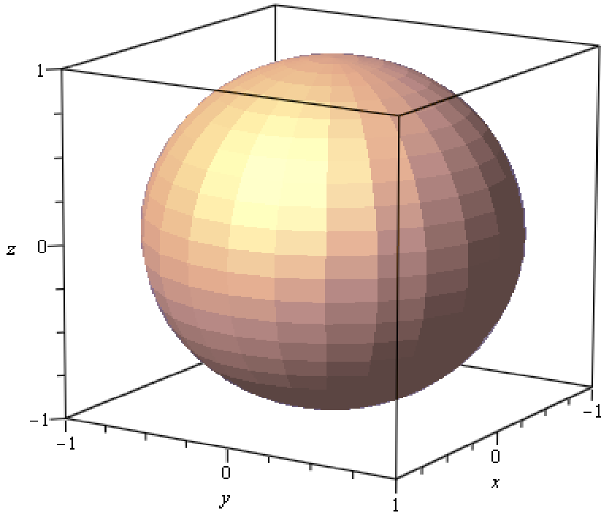
- This is bleeding edge research (like new results *last night*)
- There are still a lot of unanswered questions

Some Definitions

- $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$
- ξ is a point on the sphere

$$\xi = (\theta, \varphi) \text{ where}$$
$$\theta \in [0{,}2\pi[$$
$$\varphi \in [0, \pi]$$

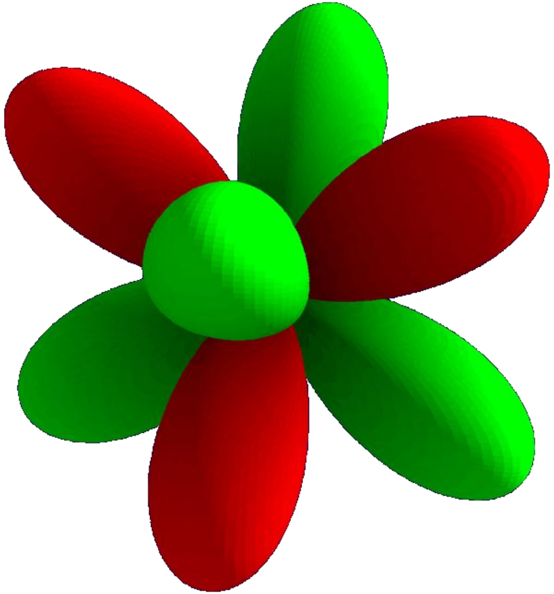
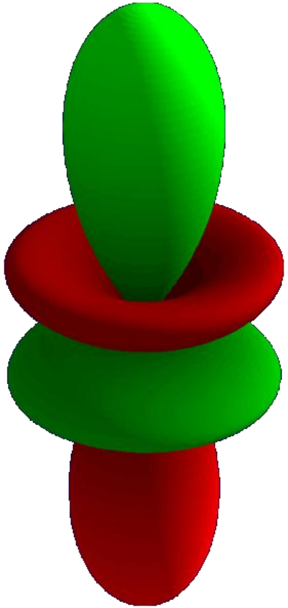
$$\xi = (x, y, z) \text{ where}$$
$$\sqrt{x^2 + y^2 + z^2} = 1$$



- Right-handed coordinate system,
  $+$ z is up

Spherical Harmonics

- The Real SH functions are a family of orthonormal basis function on the sphere.

Spherical Harmonics

- They are defined on the sphere as a signed function of every direction

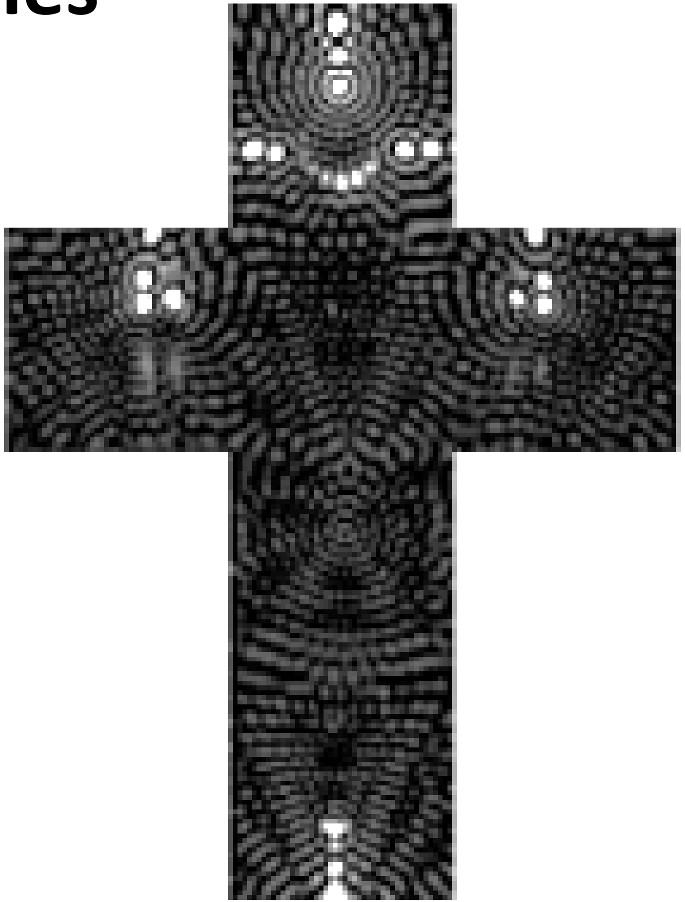
$$y_l^m(\theta,\varphi) = \begin{cases} \sqrt{2}K_l^m \cos(m\varphi) P_l^m(\cos\theta), & m > 0 \\ \sqrt{2}K_l^m \sin(-m\varphi)\, P_l^{-m}(\cos\theta), & m < 0 \\ K_l^0 P_l^0(\cos\theta), & m = 0 \end{cases}$$

- The functions are orthogonal to each other

$$\int_{\xi\in\mathbb{S}^2} y_i(\xi) y_j(\xi)\, d\xi = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & \mathrm{i} \neq j \end{cases}$$

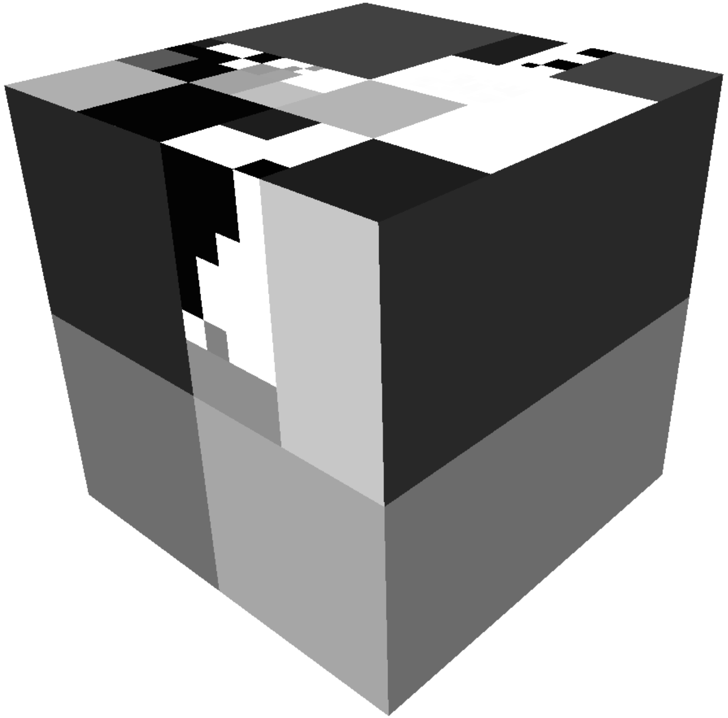
SH Deficiencies

- SH produces signed values yet all visibility functions, BRDFs and light probes are strictly positive.
- SH projections are global and smooth, visibility functions are local and sharp.
- SH reproduces a signal *at the limit*. There is no guarantee the result is close to the original at low orders. Even at high orders it "rings" esp when restricted to the hemisphere.



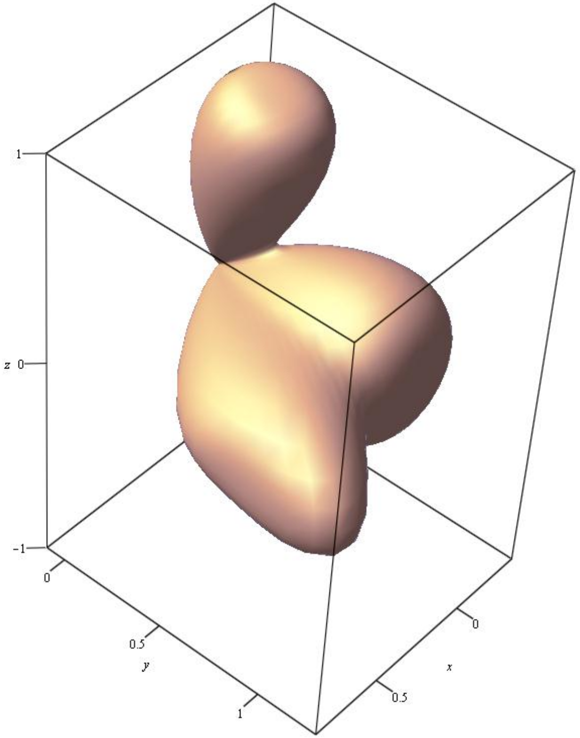
Haar Wavelets

- Haar wavelets are spatially compact and produce a lot of zero coefficients.
- Generating 6 times the coefficients, papers rely on compression and highly conditional code.
- Projecting cube faces onto the sphere introduces distortions, and seams for filtering and rotation.



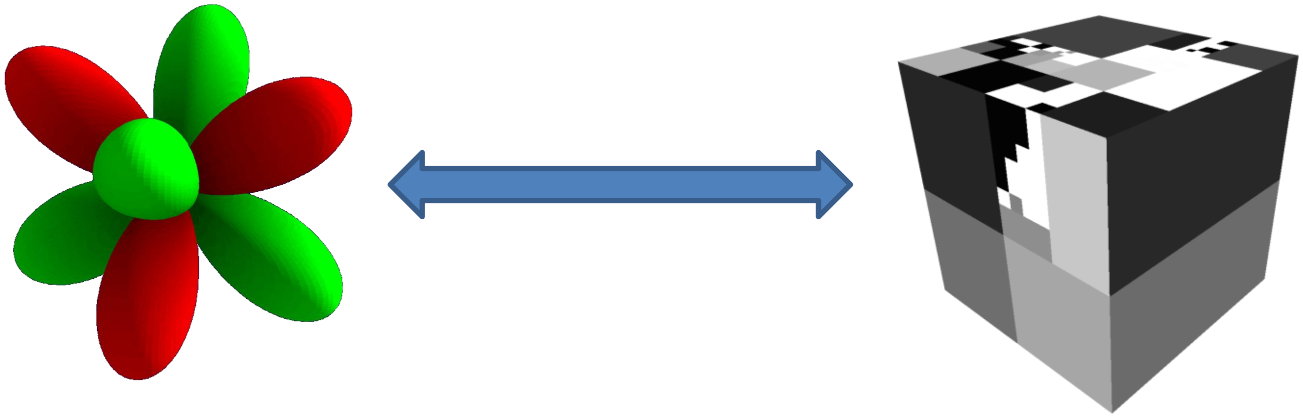
Radial Basis Functions

- Radial Basis Functions are also used, usually sums of Gaussian lobes.
- Need to solve two variables – direction and spread. Leads to conditional code that is not GPU friendly.
- *Zonal Harmonics* are another form of steerable RBF built out of orthogonal parts.



Smoothness vs. Localization

- Haar and SH are two ends of a continuum – one smooth and global, the other highly local and unsmooth. This is Spatial vs. Spectral compactness.



Q: What lives in the middle ground?

Spatial vs. Spectral

- It turns out, the Spatial vs. Spectral problem is exactly *Heisenberg's Uncertainty Principle*.
- You cannot have both spatial compactness and spectral compactness at the same time – e.g. The Fourier transform of a delta function is infinitely spread out spectrally.

- But… thanks to a theorem by David Slepian called the *Spherical Concentration Problem* you can get pretty close.

Fundamental Questions

1. Where do these Orthonormal Basis Functions come from?
2. How can we loosen the rules so we can define better functions for our own use cases?
3. What are the key properties we need to retain for our functions to be useful?

What You Need To Know

- We are going to introduce *Frame Theory* and *Spherical Quadrature*, just enough to understand two key concepts:

Parseval Tight Frames

Spherical t-Designs

Back to Fundamentals

- We choose a vector space, like $\mathbb{R}^n$ or $\mathbb{C}^n$

$$x = \{x_1, x_2, \dots, x_n\}$$

where $I = \{1, \dots, n\}$ is an index set, we say the space has a dimension n

- Using the rules of Arithmetic we can add and subtract vectors, or multiply and rescale them using a Scalar value:

$$x + y = \{x_1 + y_1, x_2 + y_2, \dots x_n + y_n\}$$

$$3x = \{3x_1, 3x_2, \dots 3x_n\}$$

Back to Fundamentals

- When we add an Inner Product and a Norm things get interesting:

$$\langle x, y \rangle = \sum_{i \in I} x_i^* y_i$$

$$|x| = \sqrt{\langle x, x \rangle}$$

- Now we can measure angles, perpendicularity, sizes, distance and similarity:

$$\langle x, y \rangle = 0 \Rightarrow x \perp y$$

- All of Geometry comes from these simple definitions

Hilbert Spaces

- A Hilbert space $\mathcal{H}$ is a vector space with a finite energy

$$\sum_{i \in \mathcal{H}} \langle e_i, e_i \rangle < \infty$$

- These *finite square summable* signals termed L^2 after Lebesgue
- L^2 is the mathematical world of data we see in the real world
  - Photographs
  - Audio streams
  - Motion Capture or GPS data

Hilbert Spaces

- The field $\mathbb{C}$ has the inner product $x\bar{y}$
- The field $\mathbb{R}^{\mathrm{n}}$ has the *dot product* defined $\sum_{i=1}^{n} x_i y_i$
- The infinite dimensional space of finite sequences $\ell_2(\mathbb{N})$ has the inner product $\sum_{i=1}^{\infty} x_i \bar{y}_i$
- The space of functions on the interval $[a, b]$ called $L^2(a, b)$ has the standard inner product:

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\, dx$$

Orthonormal Basis

- An orthonormal basis Φ for Hilbert space $\mathcal{H}$ is a set of vectors:

$$\Phi = \{e_i\}_{i \in \mathbb{Z}}$$

where each pair of vectors are mutually orthogonal:

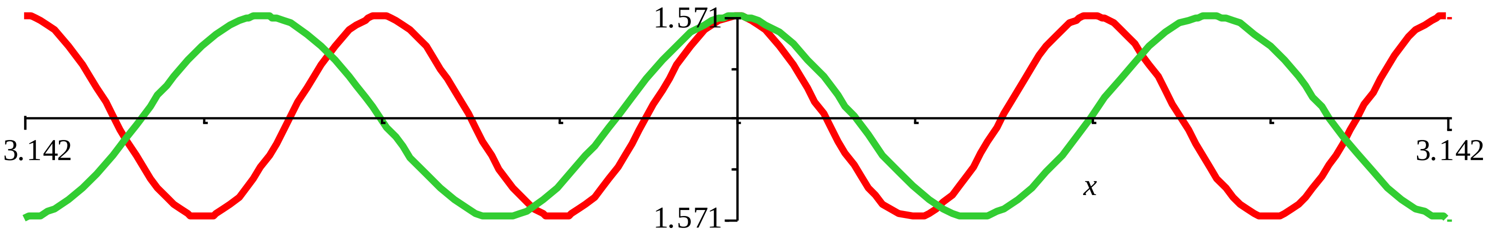
$$\langle e_j, e_k \rangle = \delta_{j,k}$$

$$\mathrm{span}(\Phi) = \mathcal{H}$$

  – A $\mathrm{span}(x)$ is the set of all finite linear combinations of the elements of x

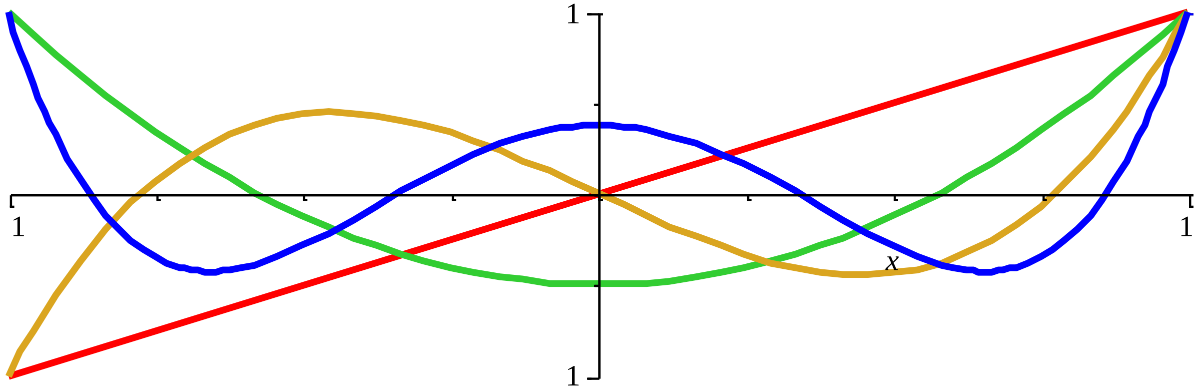
Orthonormal Bases

- For example
  - the family $\left\{\frac{1}{2\pi}e^{inx}\right\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(-\pi,\pi)$ called the *standard Fourier basis* from which we get the Fourier transform.



Orthonormal Bases

- For example
  - The family of polynomials $\left\{1, x, x^2 - \frac{1}{3}, x^3 - \frac{3}{5}x, \ldots\right\}$ are the *Legendre Polynomials*, and form an orthonormal basis on the interval $L^2(-1,1)$



Orthonormal Bases

- For example
  - The family $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis on $\ell^2(\mathbb{N})$ where

$$e_1 = \{1,0,0,0,0,0,0,\dots\}$$
$$e_2 = \{0,1,0,0,0,0,0,\dots\}$$
$$e_3 = \{0,0,1,0,0,0,0,\dots\}$$

  - $\ell^2(\mathbb{N})$ is the infinite dimensional space of finite, time-related signals like audio, motion capture joints or accelerometer data.

Orthonormal Basis Characteristics

- **Projection:** Given a signal or function $f \in \mathcal{H}$

$$c_i = \langle e_i, f \rangle$$

- If e_i is a vector, this projection is a dot product.
  If e_i is a function in 1D this is an integral $\int_a^b e_i(x) f(x) dx$
  If e_i is a function on the sphere, this integral is over the sphere $\mathbb{S}$

$$\int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} e_i(\theta, \varphi) f(\theta, \varphi) \sin \varphi \, d\theta \, d\varphi$$

Orthonormal Basis Characteristics

- **Perfect reconstruction**:

$$f = \sum_{i \in I} \langle e_i, f \rangle e_i \qquad \text{for all } f \in \mathcal{H}$$

- This says we can project then exactly reconstruct our signal from just it’s coefficients

Orthonormal Basis Characteristics

- **Parseval's Identity:**

$$\|f\|^2 = \sum_{i \in I} |\langle e_i, f \rangle|^2 \qquad \text{for all } f \in \mathcal{H}$$

- Sometimes called *norm preservation*, this says that the total energy in the function is the same as the magnitude of the coefficients.
  - This is a key property for a lot of algorithms. Working on coefficients is a lot quicker than working on functions.

ONB Characteristics

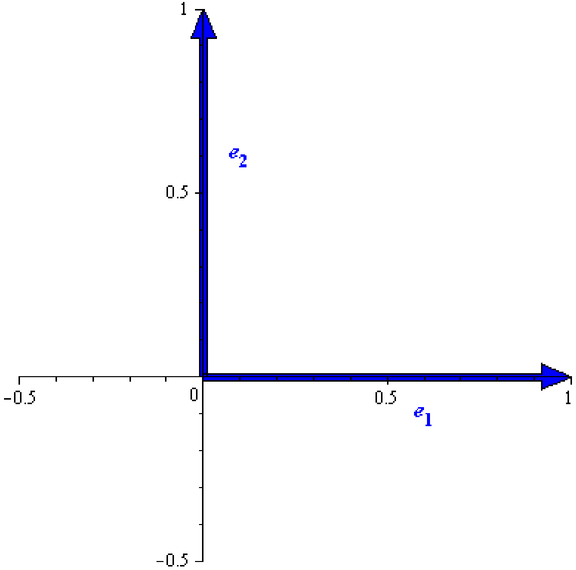
- **Successive Approximation:**

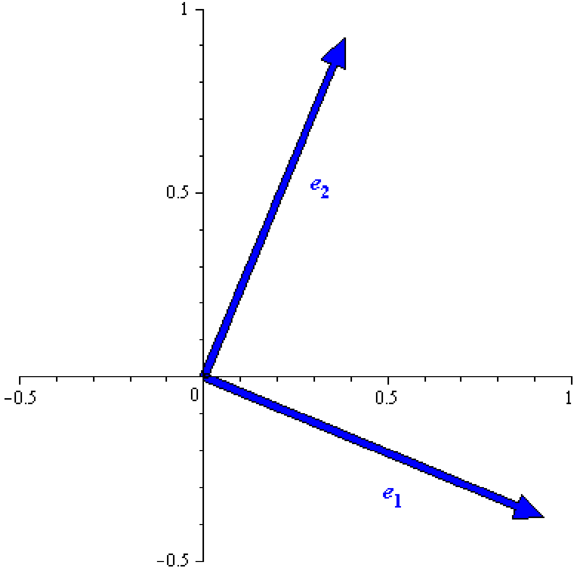
$$\hat{x}^{(k+1)}=\hat{x}^{(k)} + \langle e_{k+1}, x\rangle e_{k+1}$$

- This is a roundabout way of saying that projecting to a subset of indexes is the best approximation in a *least squares sense*.

General Bases

- We use Orthonormal Bases all the time
- Every rotation matrix in 3D is an Orthonormal Basis





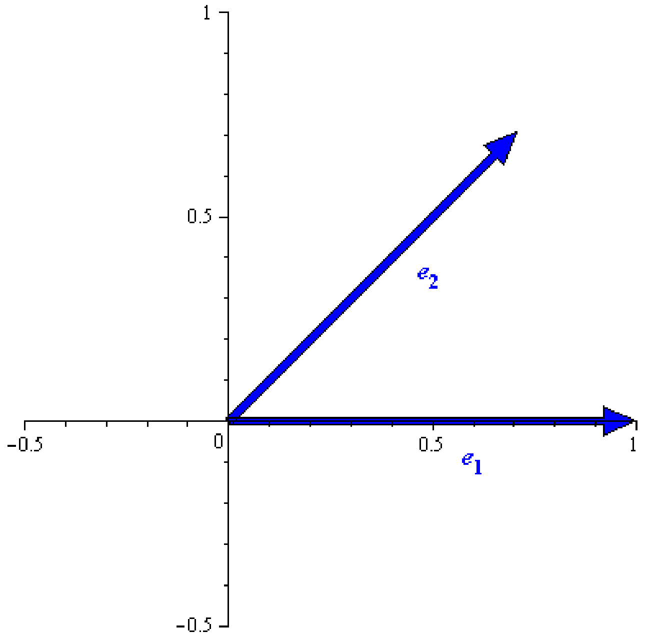
General Bases

- What if you chose vectors that are not orthogonal?

$$\Phi = \{e_1, e_2\}$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$



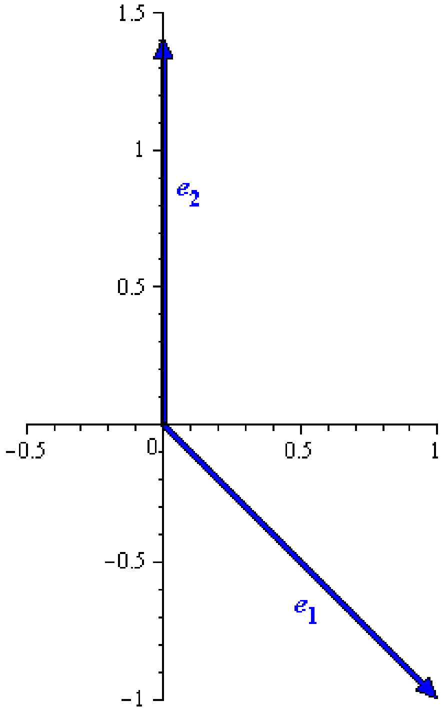
General Base

- We can still represent points, but we need a "helper" basis to get us there.

$$\widetilde{\Phi} = \{\tilde{e}_1, \tilde{e}_2\}$$

$$\tilde{e}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\tilde{e}_2 = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$$



General Bases

- We can now project the point $f = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

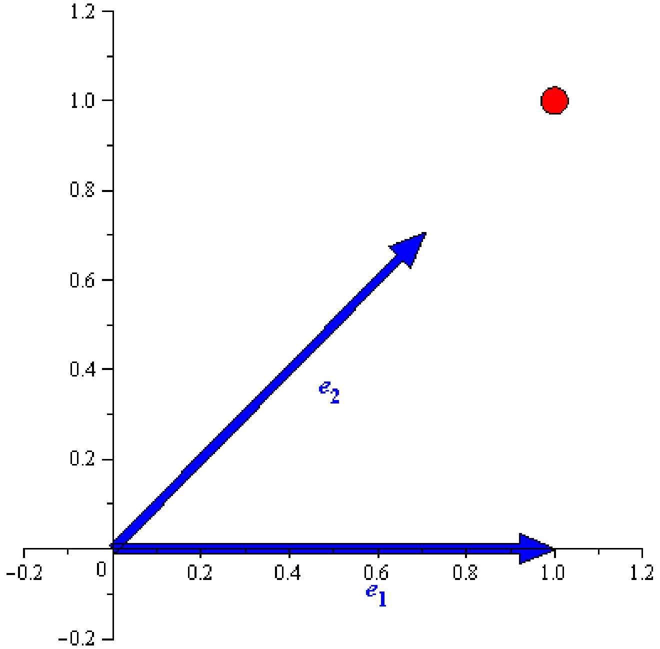
$$f' = \sum_{i=1}^{2} \langle \tilde{e}_i, f \rangle e_i$$

$$= \langle \tilde{e}_1, f \rangle e_1 + \langle \tilde{e}_2, f \rangle e_2$$

$$= (1 \cdot 1 + -1 \cdot 1) e_1 + \left(0 \cdot 1 + \sqrt{2} \cdot 1\right) e_2$$

$$= 0 \cdot e_1 + \sqrt{2} \cdot e_2$$

$$= \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$$



Biorthogonal Bases

- This second "helper" matrix is called the *dual basis* $\widetilde{\Phi}$

$$\langle e_1, \tilde{e}_1 \rangle = 1 \cdot 1 + 0 \cdot -1 = 1$$
$$\langle e_2, \tilde{e}_2 \rangle = \frac{\sqrt{2}}{2} \cdot 0 + \frac{\sqrt{2}}{2} \cdot \sqrt{2} = 1$$
$$\left\langle e_j, \tilde{e}_k \right\rangle = \delta_{j-k} \quad where\ \delta =$$

- *Biorthogonal bases* are pairwise orthogonal and commute.

$$f = \sum_{i \in I} \langle \tilde{e}_i, f \rangle e_i = \sum_{i \in I} \langle e_i, f \rangle \tilde{e}_i$$

Matrix Notation

- Now we switch to a matrix notation.

$$\Phi = \{e_1, e_2, e_3, \dots\}$$

- Every basis in $\mathcal{H}$ can be written as a matrix with basis vectors as columns

$$= \begin{bmatrix} e_{1x} & e_{1y} \\ e_{2x} & e_{2y} \\ \vdots & \vdots \end{bmatrix}$$

- Points are now column vectors.

$$p = \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix Notation

- Our projection and reconstruction now turn into *operators*

$$p = \widetilde{\Phi} f$$

$$f = \Phi^* p$$

(where M^* is the transpose)

- We can now show that orthonormal bases are *self dual*:

$$\widetilde{\Phi} = \Phi$$
$$\widetilde{\Phi}\Phi^* = \mathrm{I}$$

Breaking the Rules

- What happens if we add another vector to the basis?

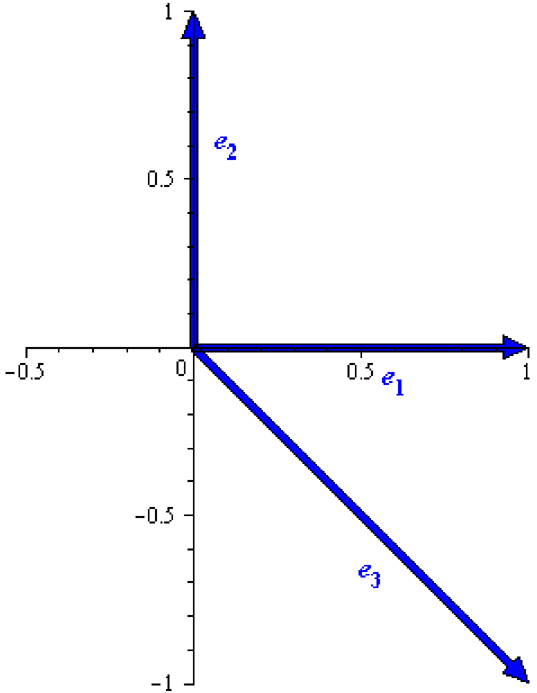
$$\Phi = \{e_1, e_2, e_3\} \qquad \widetilde{\Phi} = \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \qquad = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ -1 & 0 \end{bmatrix}$$

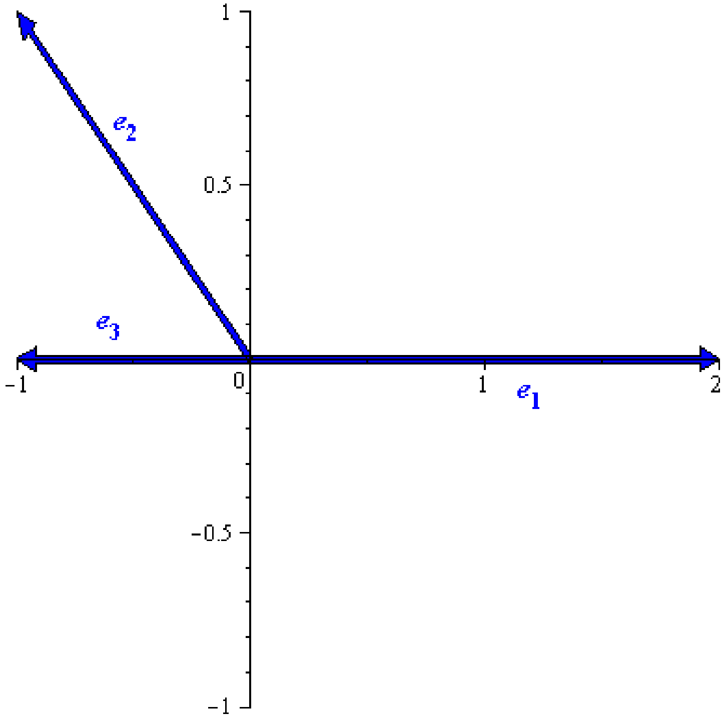
- Now we have an *overcomplete system*, and coordinates are now linearly dependent

Breaking the Rules

$\Phi = \{e_i\}_{i \in I}$



$\widetilde{\Phi} = \{\tilde{e}_i\}_{i \in I}$



Breaking the Rules

- We can still project a point and reconstruct it

$$p = \widetilde{\Phi} f$$

$$= \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$$f = \Phi^* p$$

$$= \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

General Biorthogonal Bases

- Biorthogonal bases demonstrate *Perfect Reconstruction* but we lose *Norm Preservation* and *Successive Approximation*

$$f = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \|f\| = \sqrt{(1^2 + 1^2)} = \sqrt{2}$$

$$f' = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \qquad \|f'\| = \sqrt{(2^2 + 0^2 + (-1)^2)} = \sqrt{5}$$

Frames

- This redundant set of vectors $\Phi = \{e_i\}_{i \in I}$ is called a *frame* and the set $\widetilde{\Phi} = \{\tilde{e}_i\}_{i \in I}$ is the *dual frame*

- Just like biorthogonal bases the frame and it’s dual are interchangeable and reversible

$$\begin{aligned} f &= \Phi \widetilde{\Phi}^* f \\ &= \widetilde{\Phi} \Phi^* f \end{aligned}$$

Mercedes Benz Frame

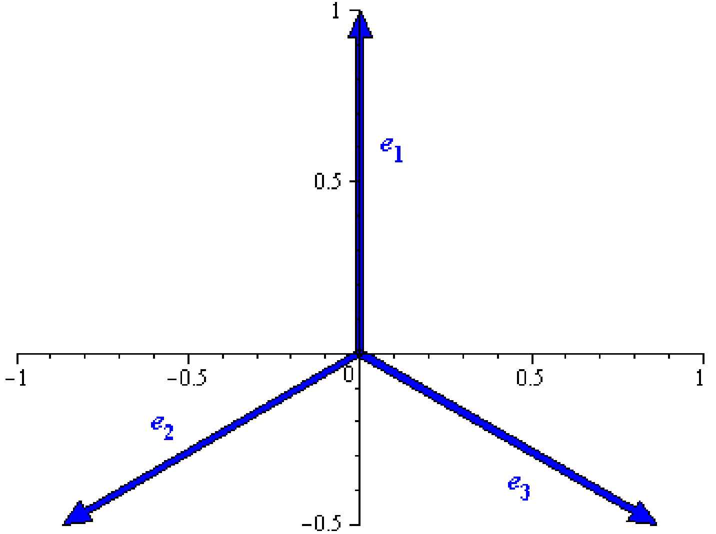
- Certain frames have properties that mimic Orthonormal bases.

- The *Mercedes Benz* frame has unit length elements and produces a norm $3/2$ times too large:

$$\sum_{i=1}^{3} |\langle e_i, p\rangle|^2 = \frac{3}{2}\|p\|^2$$

- $3/2$ is the redundancy in the system.

$$\Phi_{MB} = \begin{bmatrix} 0 & 1 \\ -\sqrt{3}/2 & -1/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$



Parseval Tight Frame

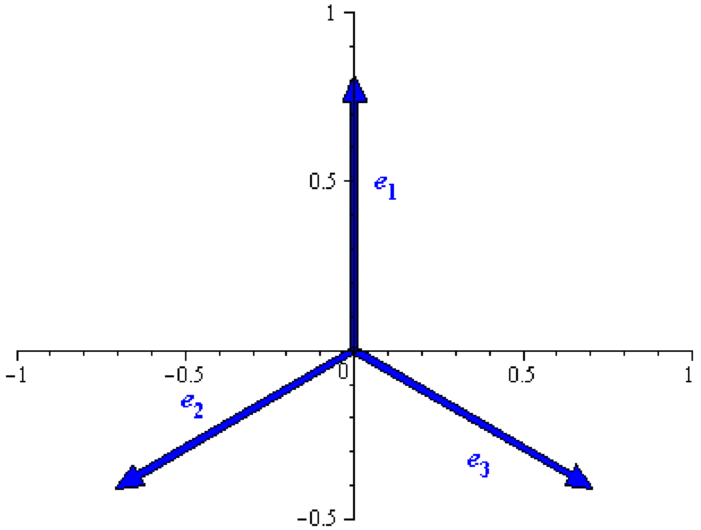
- We can factor out this constant and we end up with a frame that obeys *Parseval's identity*

$$\Phi_{PTF} = \sqrt{\frac{2}{3}}\Phi_{MB}$$

- This is called a *parseval tight frame*, or PTF.

- Parseval tight frames have all the same properties as orthonormal bases, except for *successive approximation*.

$$\Phi_{PTF} = \begin{bmatrix} 0 & \sqrt{2/3} \\ -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$



PTF-Mercedes Benz is Self Dual

- The PTF-MB basis is self dual and preserves the norm.

$$\Phi_{PTF} f = \begin{bmatrix} 0 & \sqrt{2/3} \\ -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.8165 \\ -1.1154 \\ 0.2989 \end{bmatrix} = f'$$

$$\Phi_{PTF}^{*} f' = \begin{bmatrix} 0 & -1/\sqrt{2} & \frac{1}{\sqrt{2}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0.8165 \\ -1.1154 \\ 0.2989 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = f$$

$$\|f\| = \sqrt{2} \qquad \|f'\| = 1.4142$$

Parseval Tight Frame

- PTFs have *exact reconstruction* like orthonormal bases
- PTFs are *self dual*, so we do not need a *dual frame* to project

$$f = \sum_{i=1}^{n} \langle e_i, f \rangle e_i$$

Frame Bounds

- A family of elements $\{e_n\}_{n\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ is a *frame* if there exists positive constants A and B such that:

$$A\|f\|^2 \leq \sum_{n\in\mathbb{Z}} |\langle e_n, f\rangle|^2 \leq B\|f\|^2$$

- The two values A and B are called the *frame bounds*
- Ensuring $A > 0$ means that the whole space is spanned
- Ensuring $B < \infty$ means the space is finite

Frame Bounds

- We can categorize frames based on their construction

$\|e_i\| = 1$	Unit Frame
$A = B$	Tight Frame
$A = B = 1$	Parseval Tight Frame

- Any tight frame can be factored into a PTF

Gram Matrix

- One way to check that a frame is a tight frame is to generate the *Gram Matrix* $\Phi\Phi^*$

$$M_{ij} = \langle e_i, e_j \rangle$$

- If the frame is Parseval Tight, it will have 1 in the leading diagonal and the frame bound A in the off-diagonals

$$\Phi = \{e_1, e_2, e_3, e_4\}$$

$$\mathrm{M} = \Phi\Phi^* = \begin{bmatrix} 1 & a & a & a \\ a & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{bmatrix}$$

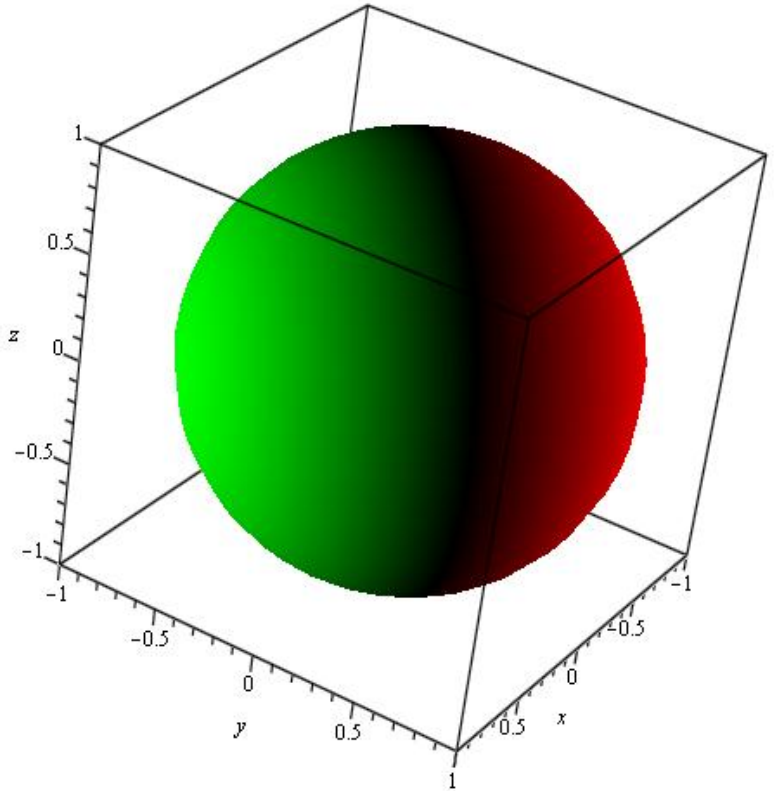
Spherical Polynomials

- A spherical polynomial is simply an expression in (x, y, z) that is evaluated on the surface of the unit sphere.

- Add the highest power on each axis to find the *order* of the polynomial, e.g.

$$f(x, y, z) = 3x^2 + yz$$

is a $2 + 1 + 1$ = 4th order spherical polynomial



Integrating on the Sphere

- We have three ways of integrating over a sphere

1. Symbolic integration over $\mathbb{S}^2$

$$\int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} e_i(\theta,\varphi) f(\theta,\varphi) \sin\theta \, d\theta \, d\varphi$$

2. Numerical integration using unbiased Monte Carlo

$$E(f) \approx \frac{4\pi}{N} \sum_{n=1}^{N} e_i(\xi_n) f(\xi_n)$$

Gaussian Quadrature

- If you are integrating a fixed order polynomial over a closed range, Gaussian quadrature can find the integral using a small number of evaluations

Simpson’s rule graph

- Trapezium Rule is a quadrature for linear curves.
- Simpson’s Rule is a quadrature for quadratic curves.

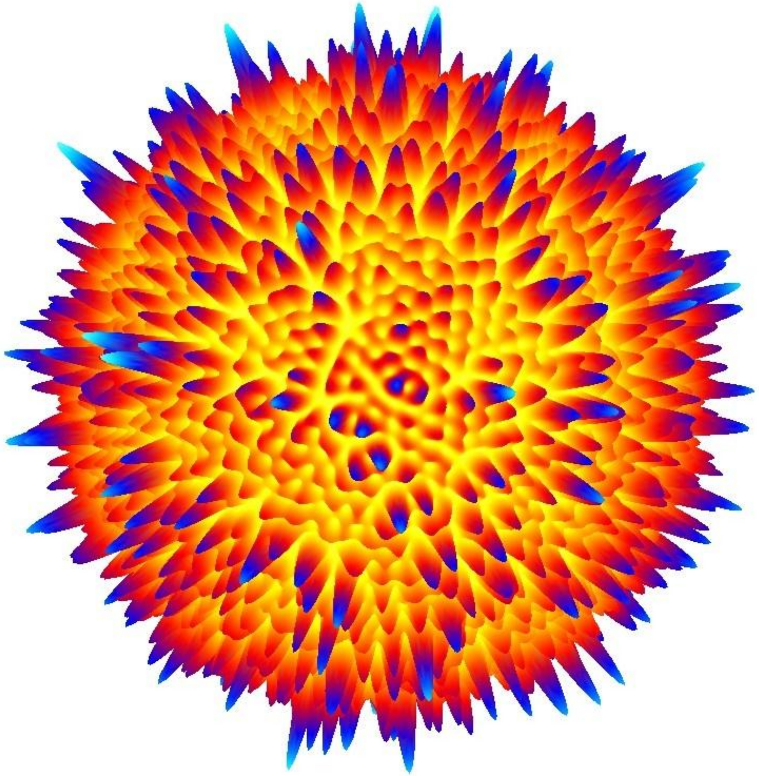
Spherical Quadrature

- Given a set of points and their weights, quadrature will quickly find you the integral

$$\int_{-1}^{1} f(x)dx = \sum_{j=1}^{N} w_j f(x_j)$$

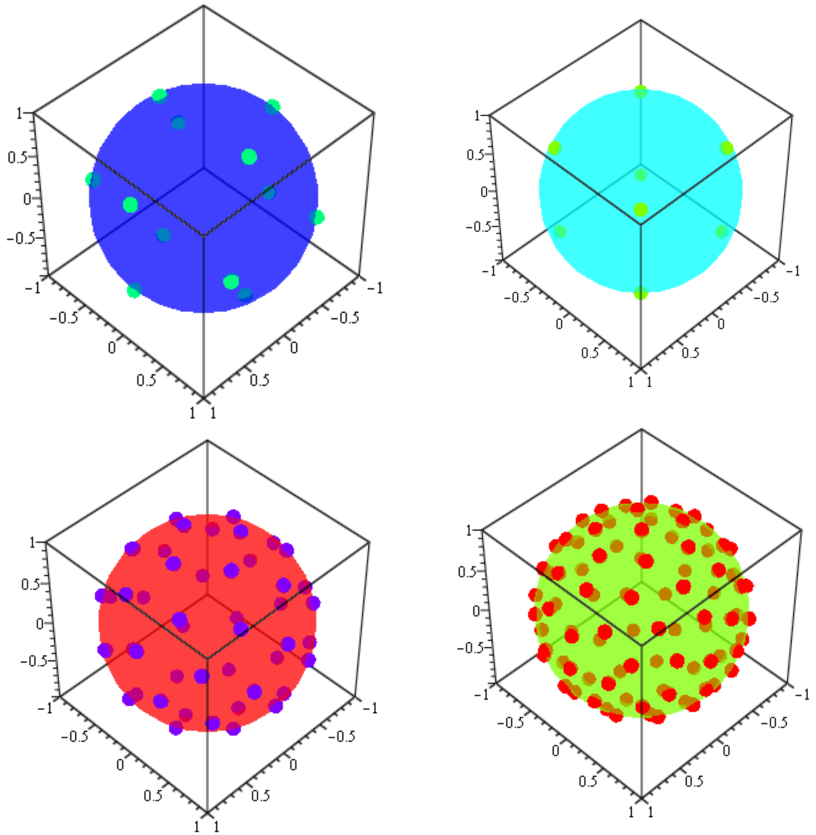
  - To find the integral over $[a, b]$ we scale the range on x_j

- This also applies to integration over the sphere, sometimes termed *spherical cubature*

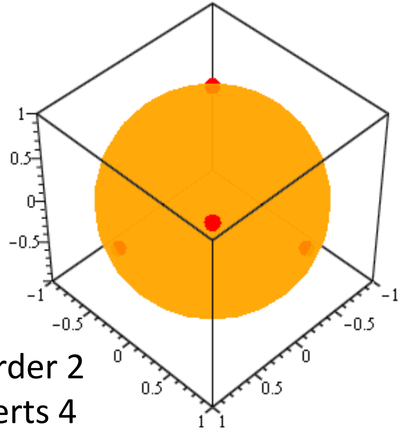


Spherical t-designs

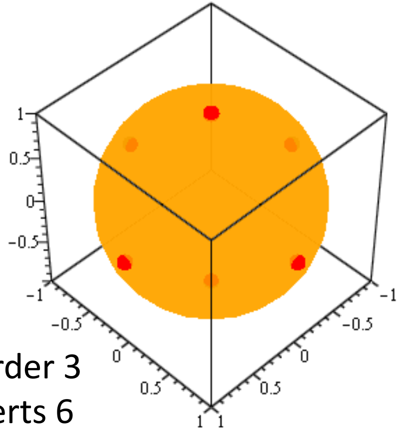
- A *spherical t-design* is a special quadrature on the sphere where each point has the same weight $^1/_N$
- There are designs in 3D for N points from 1 to 100, the full list of known low order designs is on the web.
- A t-design can accurately integrate a spherical polynomial of order t *and below*.



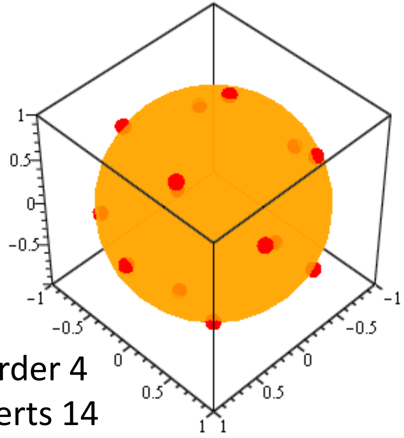
Minimum Order t-designs



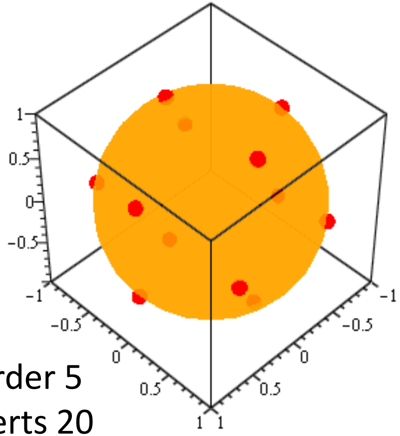
order 2
verts 4



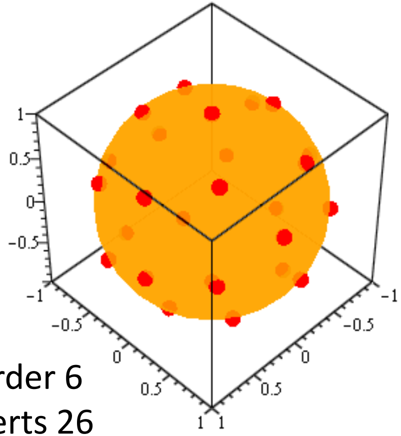
order 3
verts 6



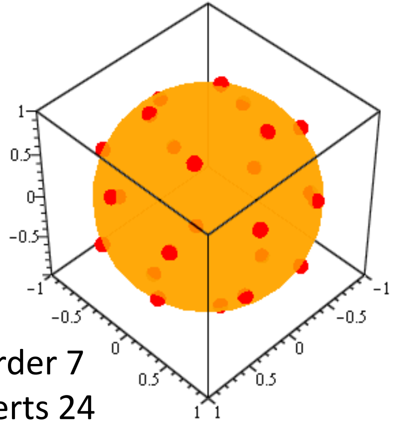
order 4
verts 14



order 5
verts 20



order 6
verts 26



order 7
verts 24

The Mission

- We need to find a spherical basis that is
  - Is defined natively on the sphere
  - Retains the norm as a Parseval Tight Frame
  - Allows us to select the number of coefficients
  - Is spectrally and spatially concentrated
  - Is cheap to project
  - Is cheap to rotate
  - Exhibits rotational invariance

Spherical Needlet

- Thanks to Narcowitch et al, 2005 we have the *Spherical Needlet,* a type of third generation Wavelet

$$e_i(\xi) = \sqrt{\lambda_i} \sum_{\ell=0}^{d} b\left(\frac{\ell}{B^j}\right) \sum_{m=-\ell}^{\ell} \bar{Y}_{\ell m}(\xi) Y_{\ell m}(\xi_i)$$

Where $Y_{\ell m}(\xi)$ are the complex Spherical Harmonics, B is the *bandwidth* and j is the polynomial order

Simplifications

- The product-sum of all Complex Spherical Harmonics in one "row" is just a simple Legendre polynomial:

$$\frac{2n+1}{4\pi} P_\ell(\xi' \cdot \xi) = \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\xi) Y_{\ell m}(\xi')$$

- So needlets are defined in frequency space from orthonormal parts and are natively embedded on the sphere

Legendre Polynomials

- The Legendre polys are normalized to simplify the definitions.

$$L_\ell(\xi' \cdot \xi) = \frac{2n+1}{4\pi} P_\ell(\xi' \cdot \xi)$$

- Legendre polys can be quickly generated iteratively using Bonnet's Recursion:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

$$\text{where} \quad \begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \end{aligned}$$

Littlewood-Paley Decomposition

- The key part of the algorithm is the $b\left(\frac{\ell}{B^j}\right)$ function.

$$f(t) = \begin{cases} \exp\left(-\frac{1}{1-t^2}\right), & -1 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$w(u) = \frac{\int_{-1}^{u} f(t)dt}{\int_{-1}^{1} f(t)dt}$$

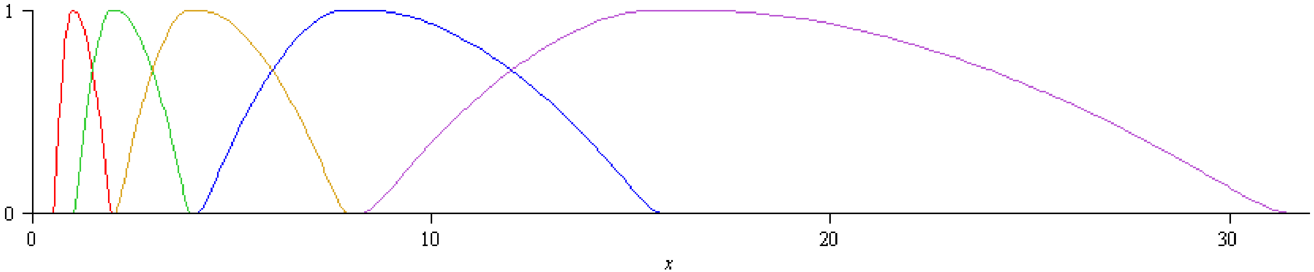
$$p(t) = \begin{cases} 1, & 0 \le t \le \frac{1}{B} \\ w\left(1 - \frac{2B}{B-1}\left(t - \frac{1}{B}\right)\right), & \frac{1}{B} \le t \le 1 \\ 0, & t > 1 \end{cases}$$

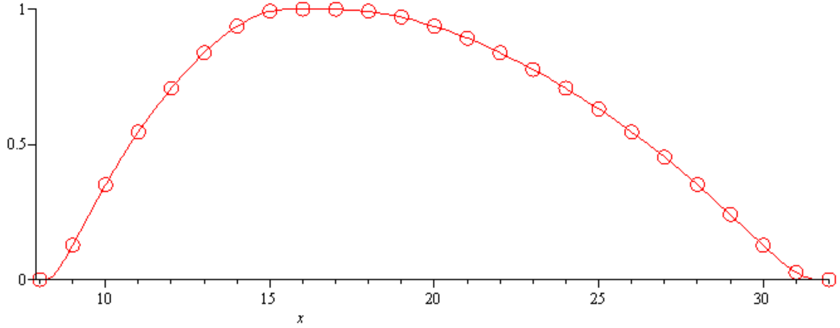
$$b(t) = \sqrt{p\left(\frac{t}{B}\right) - p(t)}$$

- Defined as a continuous function, evaluated at integer points.

Littlewood Paley Decomposition

- LP Decomposition allows us to break down spectral space into chunks of bandwidth B.





Spherical Needlet

- For use in signal space, the needlet is defined as:

$$e_i(\xi) = \sqrt{\lambda_i} \sum_{\ell=0}^{d} b\left(\frac{\ell}{B^j}\right) L_\ell(\xi \cdot \xi_i)$$

A single needlet

over the sphere

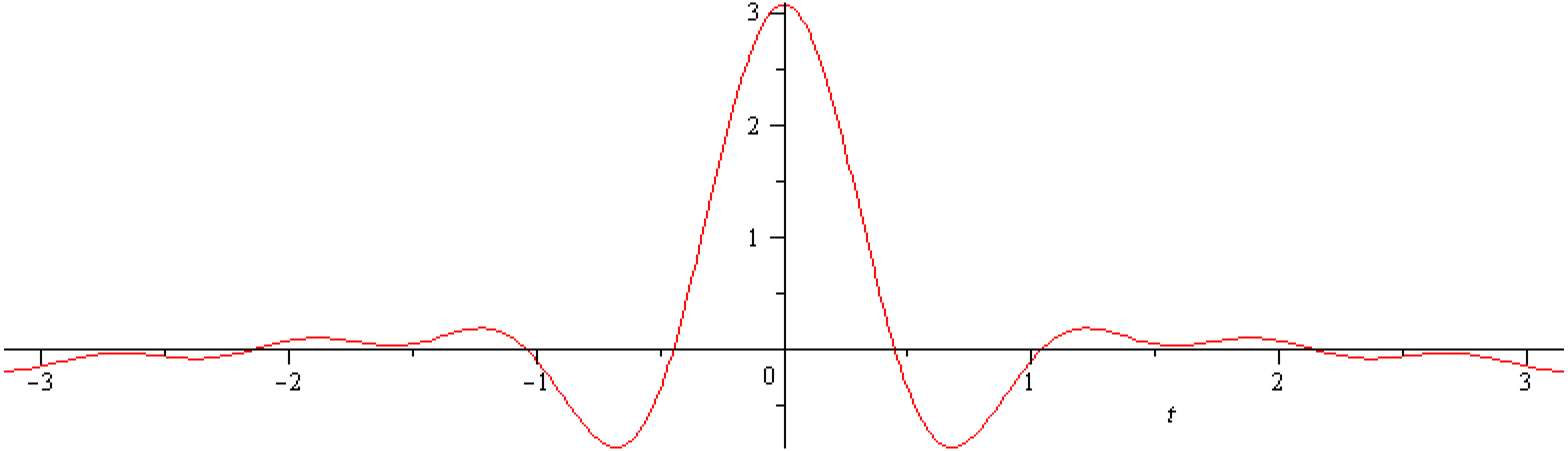
quadrature weight

Littlewood-Paley weighting

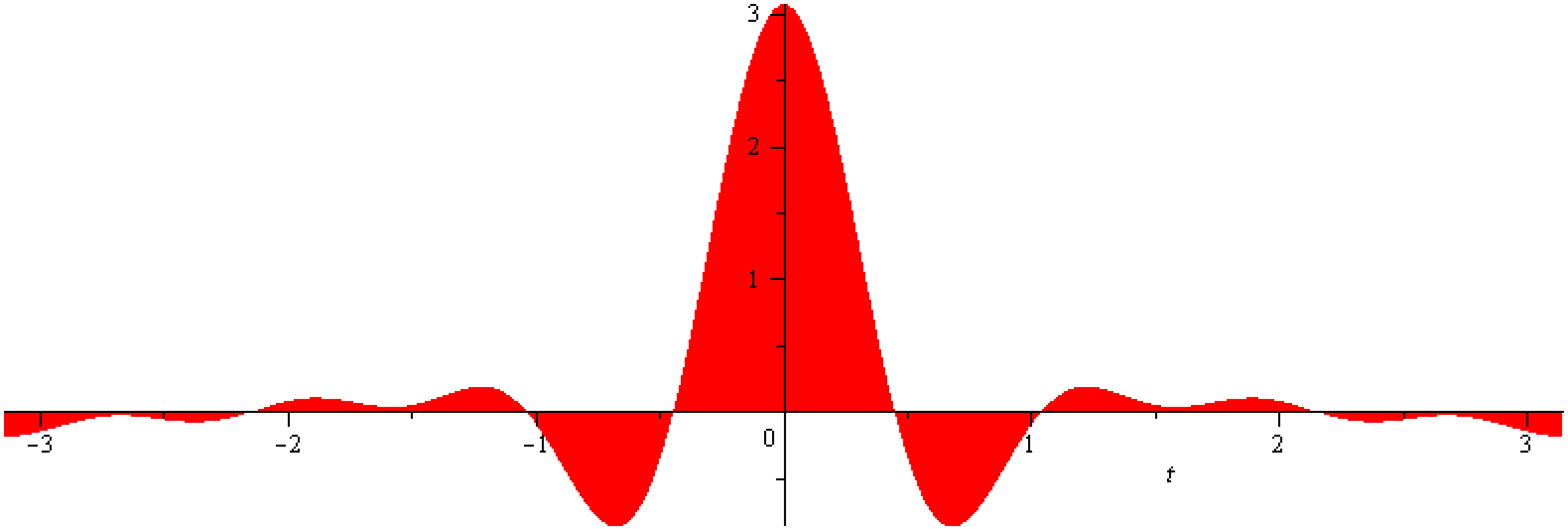
Legendre polynomial

quadrature direction

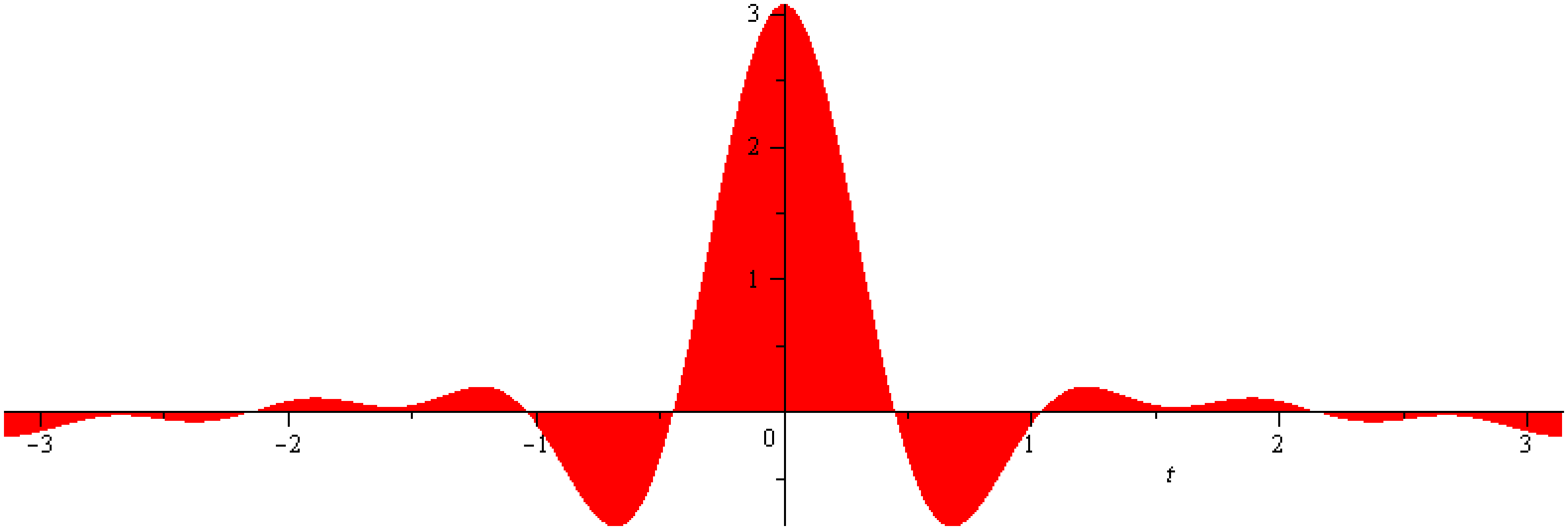
Spherical Needlet



What does this integrate to?

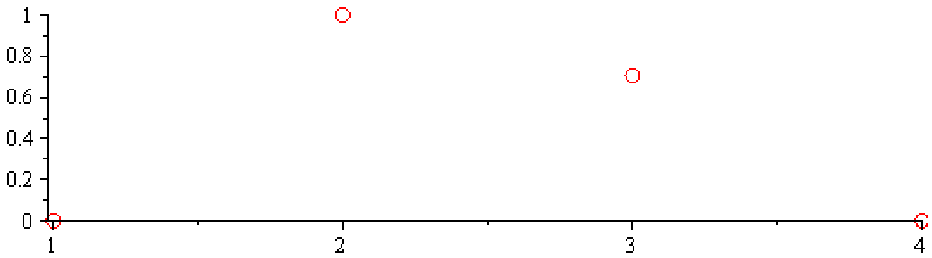


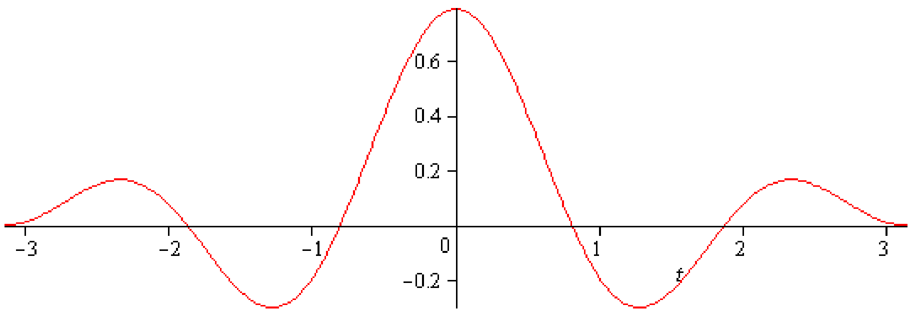
What does this integrate to?

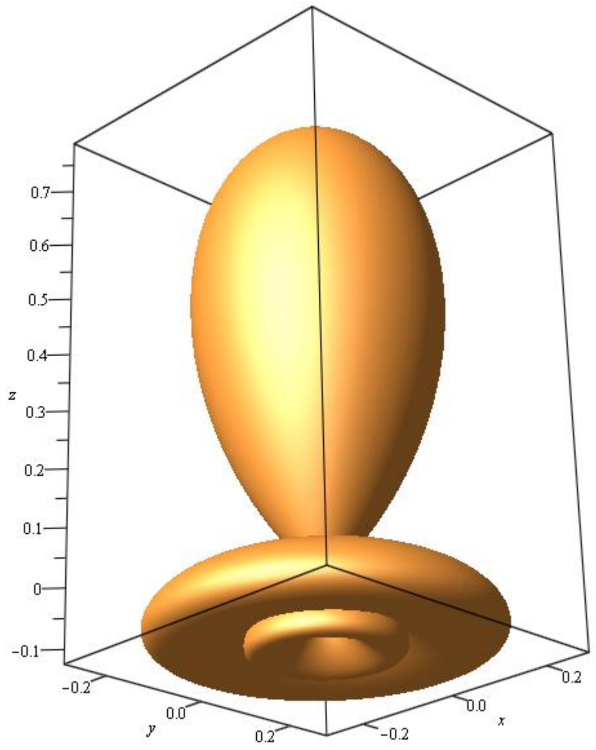


$$\int_{\xi \in \mathbb{S}} e_i(\xi) d\xi = 0$$

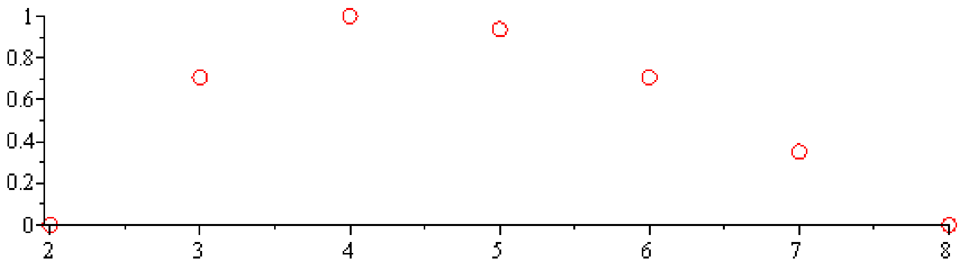
Needlet B=2.0 and j=1

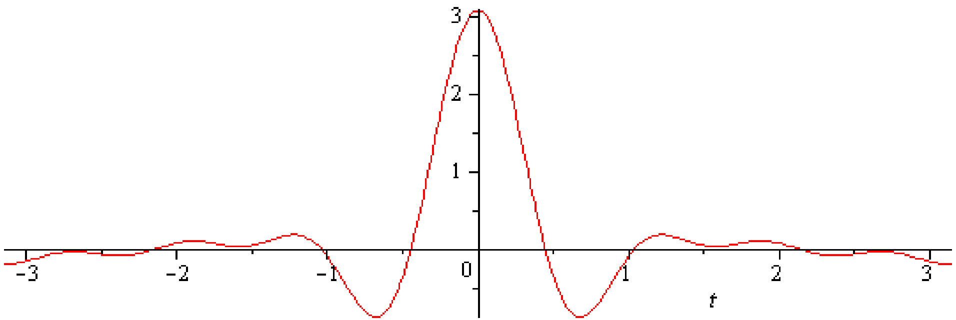


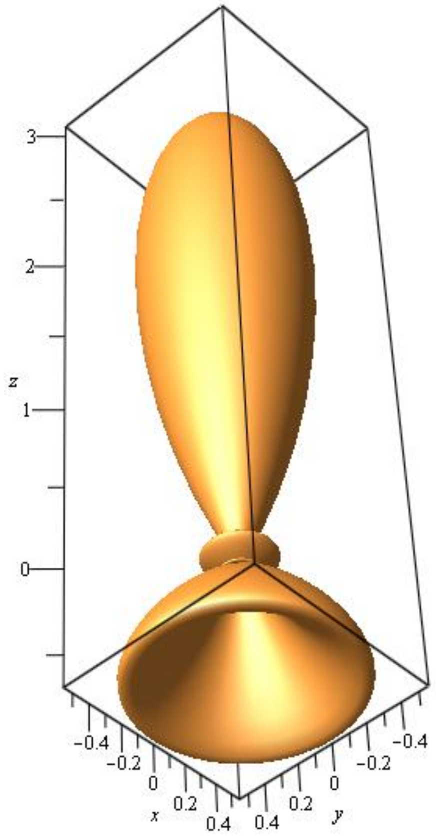




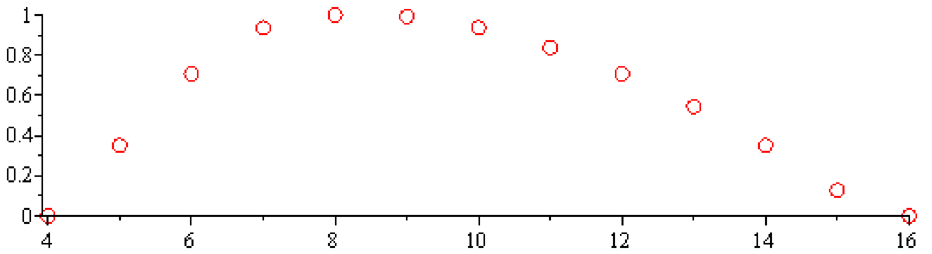
Needlet B=2.0 and j=2

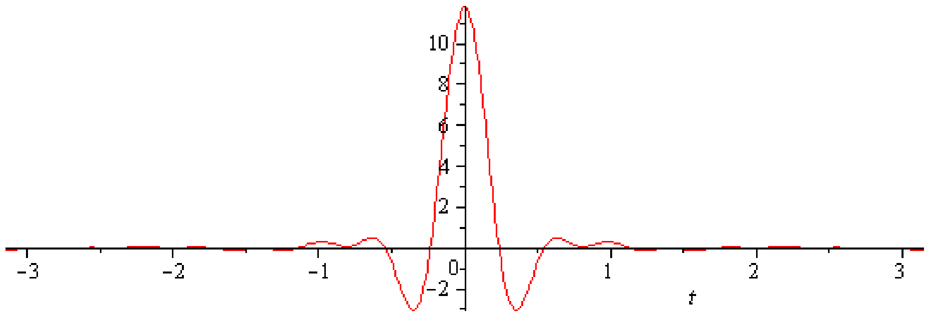


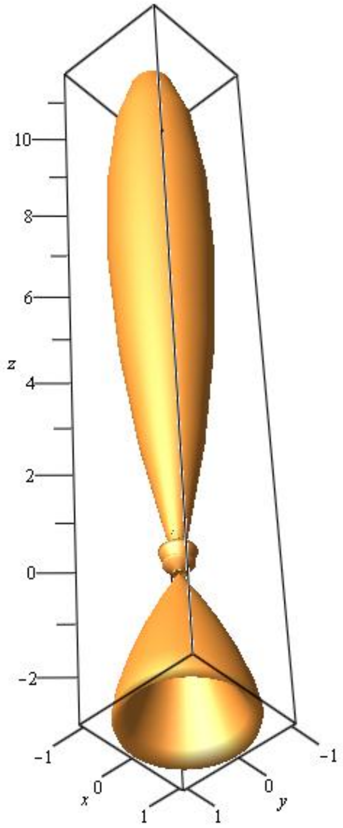




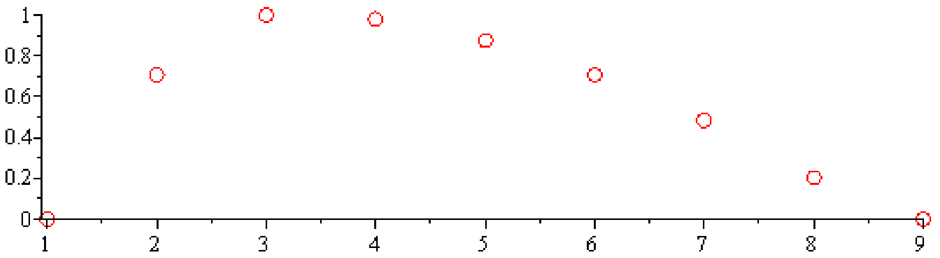
Needlet B=2.0 and j=3

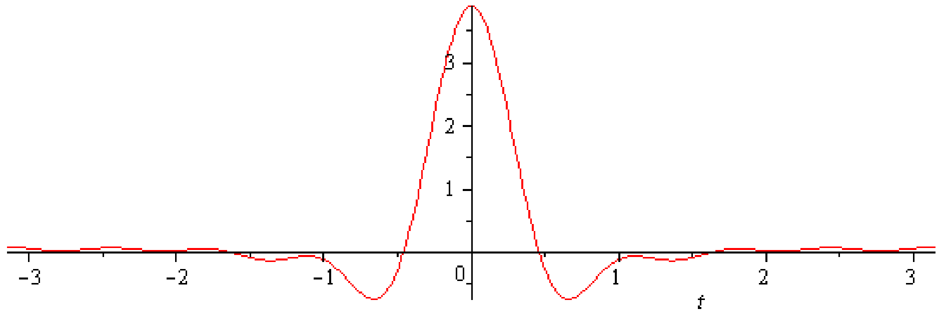


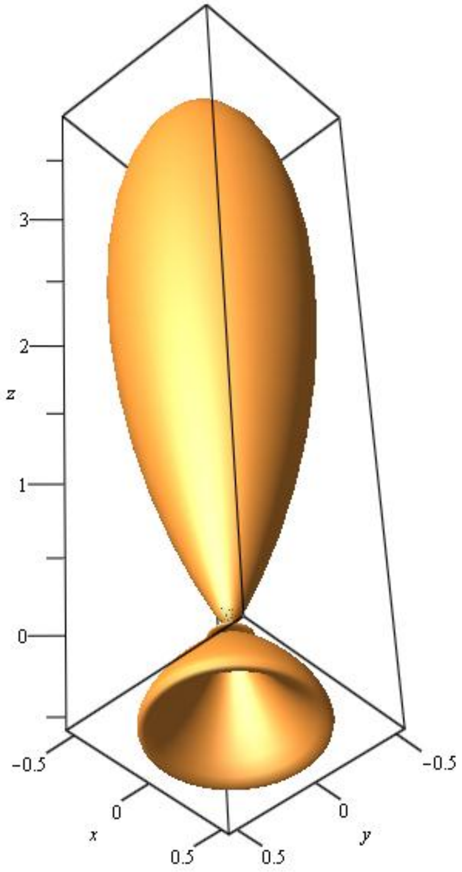




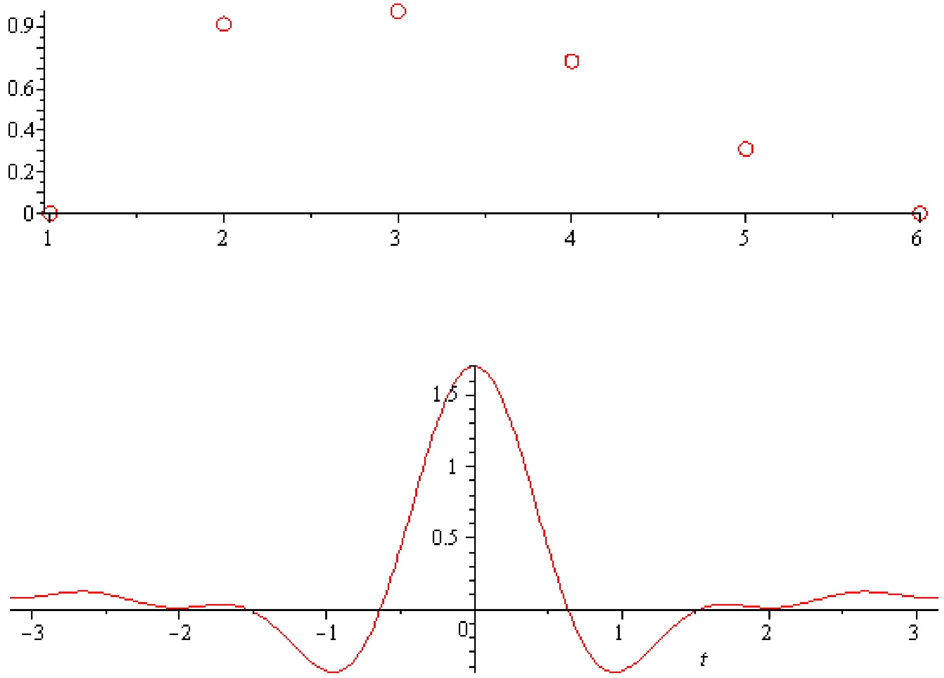
Needlet B=3.0 and j=1

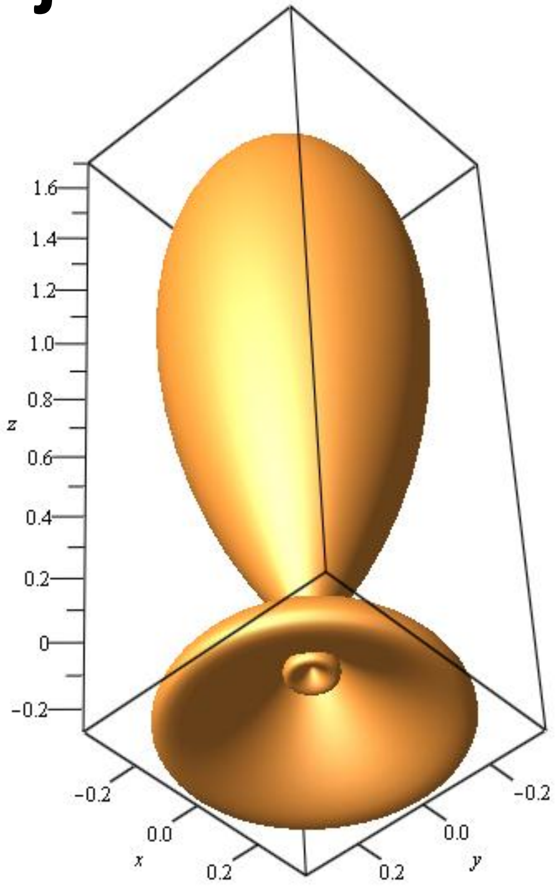






Needlet B=2.4 and j=1





Spherical Basis

- The complete spherical basis is a set of needlets, each pointing in a quadrature direction

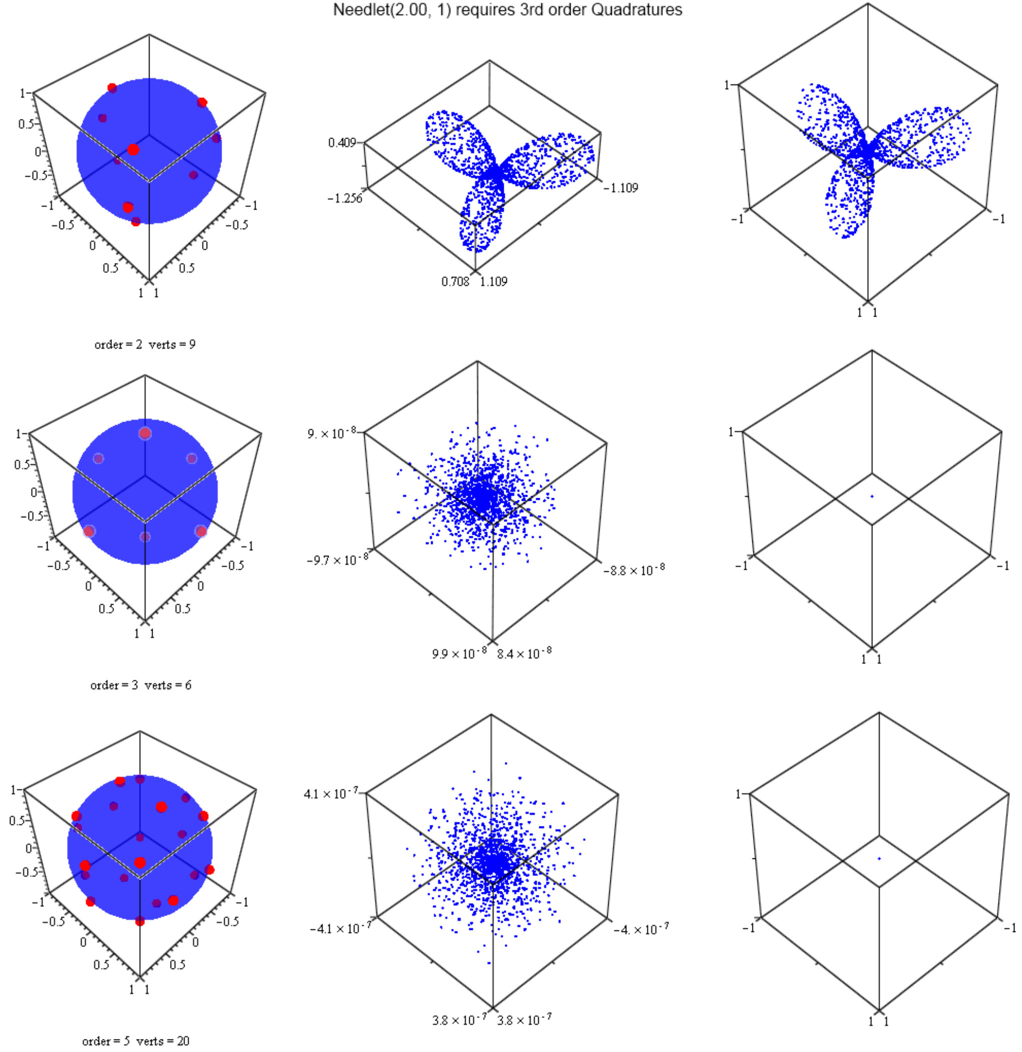
$$\Phi = \{e_i\}_{i \in (1,N)}$$

1. Needlets are a solution to the *Spherical Concentration Problem*
   - for a given bandwidth it is the most compact spatial support

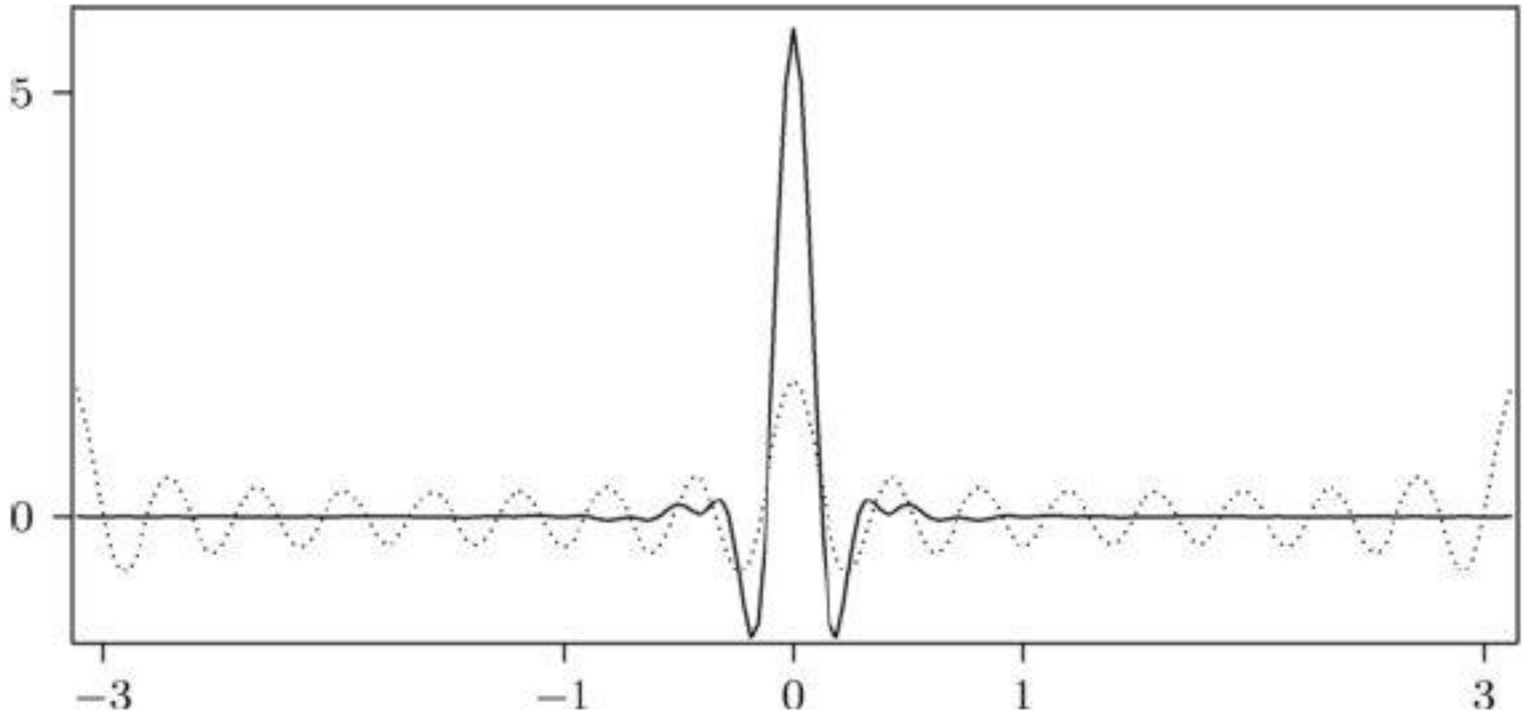
2. The sum of needlet bases over $j = \{2,3,4,\dots\}$ form a tight frame on the sphere.

3. A needlet of order N can exactly reconstruct spherical polynomials of order N and below.

Needlet(2.00, 1) requires 3rd order Quadratures

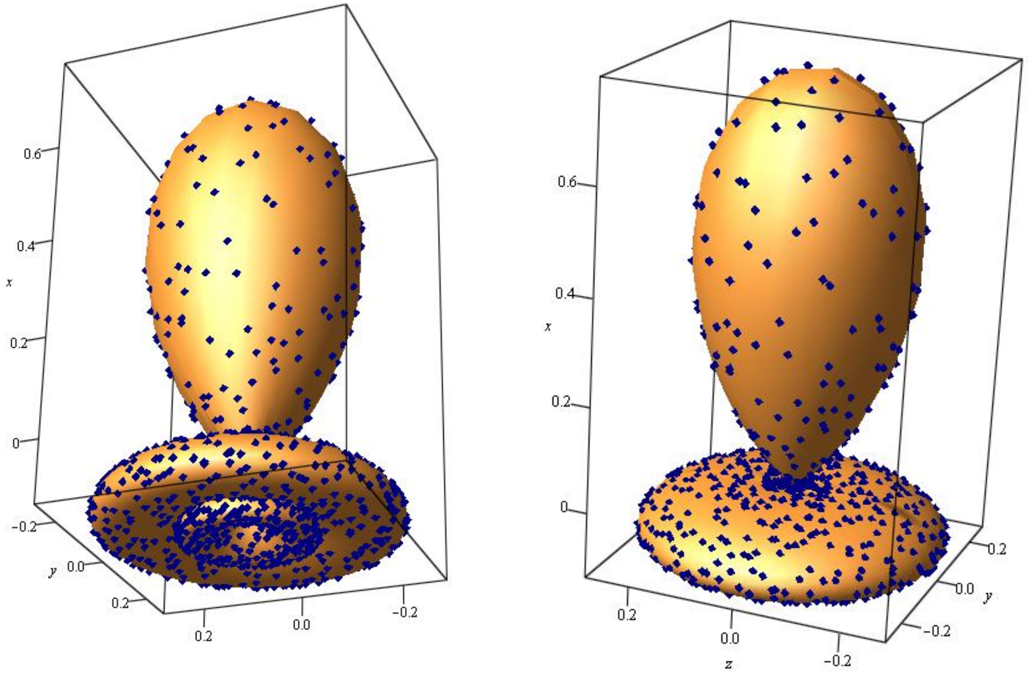


Needlet vs. SH



Monte Carlo Sampling

- Sampling needlets correctly requires non-uniform sampling



Fast Projection

- Needlets are radially symmetric ($\xi \cdot \xi_i$ is a scalar)
- The needlet function is 1D
- Approximate the needlet with a LUT, lerp the values.

Plot error of lerp LUT versus actual function.

Fast Rotation

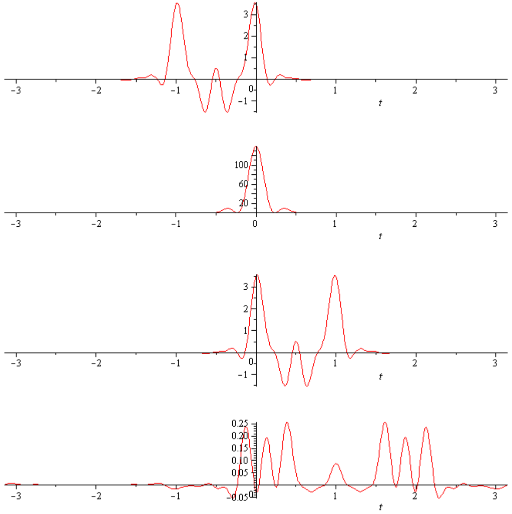
- The same rotation idea as SH, generate a matrix that reinterprets a needlet as sums of other needlets.

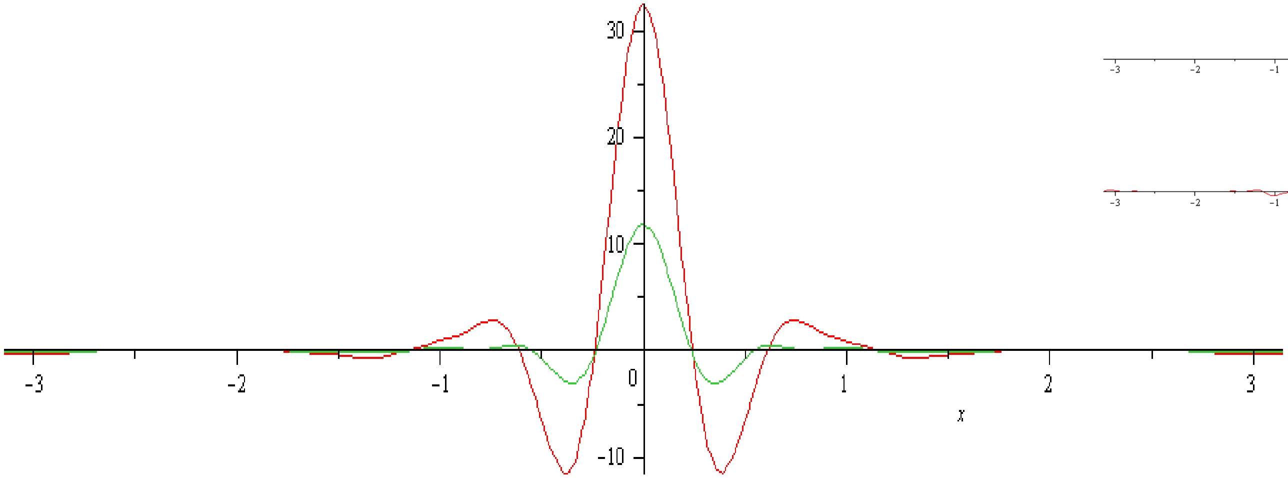
$$M_{ij} = \langle e_i, Re_j \rangle$$
$$= \int_{\xi \in \mathbb{S}} e_i(\xi) e_j\big(R(\xi)\big) d\xi$$

- The bases e_i and Re_j differ only in the quadrature direction.
- Which falls out to be a 1D function…

Fast Rotation

- By calculating each angle offset integral and tabulating it, we can generate a rotation function.





Key Features of a Spherical Basis

- Radially symmetric basis
  - Allows fast projection
  - Allows fast and stable rotation

- Defined from natively embedded atoms
  - No parameterization problems
  - Use *lifting* to construct a more performant basis
  - Spherical concentration shows that localization is possible

- Using Frames
  - Allows simpler definition of the problem
  - Who needs *successive approximation* anyway?

Future Work

- Littlewood-Paley is just one *partition of unity* optimized for spectral concentration. Other papers have optimized for spatial and other metrics.

Key References

- D. Marinucci et al, "*Spherical Needlets for CMB Data Analysis*", arxiv.org/pdf/7070.0844.pdf, 2008
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- J. Kovacevic et al, "*Life Beyond Bases: The Advent of Frames*", Signal Processing Magazine, IEEE, Vol.24, No.4, July 2007
- T. Hines, "*An Introduction to Frame Theory*", Aug 2009, http://mathpost.asu.edu/~hines/docs/090727IntroFrames.pdf

$$e_i(\xi) = \sqrt{\lambda_i} \sum_{\ell=0}^{d} b\left(\frac{\ell}{B^j}\right) L_\ell(\xi \cdot \xi_i)$$

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